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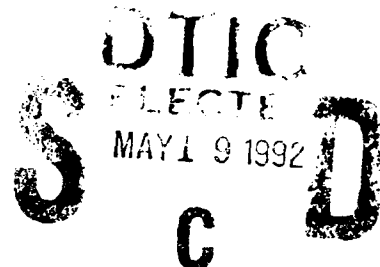
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Acoustic Target Reconstruction Using Geometrical Optics Phase Information



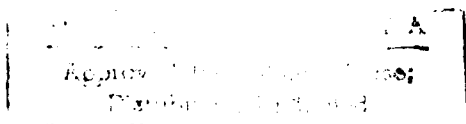
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Abstract

In this note a simple algorithm, which uses the phase information from the geometrical optics limit, to construct the shape of an object is presented. Essentially, the phase in the back scattered direction determines the equation of the tangent plane at the unique, but unknown, specular point. This plane depends upon the two spherical angles θ and ϕ , which describe the incident wave direction. The observation that the tangent plane envelopes the obstacle as θ and ϕ are varied, allows the derivation of an explicit formula for the equation of the surface in terms of the measured scattered phase. An analogous two-dimensional formula is also presented.

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I. Introduction

The field of inverse scattering is concerned with the determination of an object's shape from scattering data. In a fundamental paper [1] Keller used the theory of geometrical optics to formulate and to partially solve this problem for convex objects. Specifically, he showed that the amplitude of the scattered field in the backscattered direction was related to the curvature of the obstacle at a unique specular point in two dimensions, and to the Gaussian curvature in three dimensions. In the former case he was able to determine the obstacle's shape to within a rigid translation. He obtained similar results for the later case when the object was a surface of revolution and the incident wave propagated along the axis of rotation. In general, he observed that the problem became equivalent to Minkowski's problem whose solution required solving a nonlinear partial differential equation. Weiss used Keller's results on several specific examples and studied the effects of noisy data and of relaxing the convexity restrictions [2]. Neither author used the phase information from the geometrical optics approximation.

Majda and Ralston [3] studied the direct scattering problem and found that the argument of the determinant of the scattering matrix $s(\lambda)$ gave information about the target shape in the high frequency limit. Specifically they derived the formula

$$s(\lambda) = \frac{\lambda^3}{32\pi} V - \frac{\lambda^2}{8} S - \frac{1}{384} \int_{\partial} H \, da + O(1), \quad \lambda \rightarrow \infty$$

where λ is the wavelength of the incident radiation, V is the volume of the target, S is its surface area, H is its mean curvature, and ∂ denotes the surface of the target. Using a result of Melrose [4] they showed that $s(\lambda)$ was related to the differential cross section of the scattered field in the forward scattered direction.

In this note we give a simple algorithm which uses the phase information from the geometrical optics limit to construct the shape of a convex object. Essentially, the phase in the back scattered direction determines the equation of the tangent plane at the unique, but unknown, specular point. This plane depends upon the two spherical angles θ and ϕ , which describe the incident wave direction. By observing that the tangent plane envelopes the obstacle as θ and ϕ are varied, we obtain an explicit formula for the equation of the surface in terms of the measured scattered phase.

II. The Inverse Algorithm

A plane acoustic wave insonifies a convex target Ω which can either be soft, hard, or have a surface impedance. The incident wave vector is given by $\mathbf{V}_I = (\cos\phi \sin\theta, \sin\phi \sin\theta, -\cos\theta)^T$ where ϕ and θ are the spherical angles measured from a fixed origin O which is either inside or outside of Ω ; see Figure 1. In the high frequency limit the scattered field in the backscattered direction is given by [1,5]

$$U_s \sim \frac{A}{\sqrt{G}} \exp(-2ik\mathbf{X}_0 \cdot \mathbf{V}_R) \frac{e^{ikr}}{r}, \text{ as } r = \sqrt{x^2 + y^2 + z^2} \rightarrow \infty. \quad (1)$$

In this expression A is a constant which depends upon the particular boundary condition, \mathbf{X}_0 is the position vector of the specular point on the surface $\partial\Omega$, G is the Gaussian curvature at this point, and \mathbf{V}_R is the unit tangent vector along the specularly reflected ray. The location of the specular point is unknown.

We focus our attention on the phase term $\exp(-2ik\mathbf{X}_0 \cdot \mathbf{V}_R)$ which we assume can be measured as a function of ϕ and θ . The equation of the tangent plane at the specular point is given by

$$\mathbf{X} \cdot \mathbf{V}_R = d(\phi, \theta) \quad (2)$$

where $d(\phi, \theta) = \mathbf{X}_0 \cdot \mathbf{V}_R$ is the distance, from the fixed origin 0 to the tangent plane (see Figure 1), which is known from the phase measurements. Now equation (2) is a two parameter family of planes which envelope the surface $\partial\Omega$. To find the envelope, i.e. $\partial\Omega$, we partially differentiate (2) first with respect to ϕ and then with respect to θ . These two additional equations are then appended to (2) to give three equations in three unknowns. They are

$$x(\cos\phi \sin\theta) + y(\sin\phi \sin\theta) + z(\cos\theta) = d(\phi, \theta) \quad (3a)$$

$$-x(\sin\phi \sin\theta) + y(\cos\phi \sin\theta) = d_\phi \quad (3b)$$

$$x(\cos\phi \cos\theta) + y(\sin\phi \cos\theta) - z(\sin\theta) = d_\theta \quad (3c)$$

where the subscripts denote partial differentiation. The solution of this system of equations is

$$x = d(\cos\phi \sin\theta) - d_\phi(\sin\phi \csc\theta) + d_\theta(\cos\phi \cos\theta) \quad (4a)$$

$$y = d(\sin\phi \sin\theta) + d_\phi(\cos\phi \csc\theta) + d_\theta(\sin\phi \cos\theta) \quad (4b)$$

$$z = d(\cos\theta) - d_\theta(\sin\theta) \quad (4c)$$

which is just a parameterization of the surface $\partial\Omega$. The apparent singularity in (4) at $\theta = 0$ is removable, because $d_\phi(\phi, 0) = 0$ as can be deduced from (3b).

The determination of the phase $d(\phi, \theta)$ is carried out in the time domain because it is related to the time lag τ between the initiation of the transmitted pulse and the reception of the reflected pulse. If r is the distance from the transmitter to the origin 0, then an application of an inverse Fourier transform shows the time between the leading edges of the incident and reflected pulses to be

$$\tau = [r + 2d(\phi, \theta)]/c \quad (5)$$

where c is the acoustic wave speed in the material surrounding Ω . For smooth incident pulses without a sharp leading edges this time is difficult to

measure and is a source of error. We shall not pursue this point further at this time.

III. Examples

In the first example, let Ω be a sphere of radius R centered at the point $\mathbf{X}_C = (x_C, y_C, z_C)$ which is measured from the fixed origin O . The geometrical optics phase in the backscattered direction is $-2kd$ where d is given by

$$d(\phi, \theta) = R + \mathbf{V}_R \cdot \mathbf{X}_C = R + x_C(\sin\theta \cos\phi) + y_C(\sin\theta \sin\phi) + z_C \cos\theta. \quad (6)$$

We observe here that the phase d is not a constant, because the origin O is different from the center of the sphere. Inserting (6) into (4) we obtain

$$x = x_C + R(\cos\phi \sin\theta) \quad (7a)$$

$$y = y_C + R(\sin\phi \sin\theta) \quad (7b)$$

$$z = z_C + R \cos\theta. \quad (7c)$$

which is just the equation of our sphere.

In the second example, let Ω be the triaxial ellipsoid

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 1 \quad (8)$$

centered at the fixed origin O . Here, \mathbf{A} is a symmetric matrix whose coefficients are to be determined and $\mathbf{x}^T = (x, y, z)$. Let \mathbf{P} be the rotation matrix that diagonalizes \mathbf{A} , i.e., $\mathbf{PAP}^T = \mathbf{D}$, where \mathbf{D} is the diagonal matrix with entries $D_{ij} = \lambda_i \delta_{ij}$. The λ_i are the eigenvalues of \mathbf{A} and are the reciprocals of the axis lengths, and the columns of \mathbf{P} , \mathbf{P}_i , are the corresponding eigenvectors which geometrically point along the principal axis of the ellipsoid. In Appendix A we shown that the backscattered geometrical optics phase is given by

$$d = \sqrt{\mathbf{V}_R^T \mathbf{R} \mathbf{V}_R} \quad (9a)$$

$$\mathbf{R} = (\mathbf{eP})^T (\mathbf{eP}) \quad (9b)$$

$$\mathbf{e} = (\delta_{ij} / \sqrt{\lambda_i}) \quad (9c)$$

where the matrix \mathbf{e} is the square root of \mathbf{D}^{-1} . According to the theory presented in Section II, equation (4) gives an equivalent representation of (8) using (9). We do not demonstrate their equivalence here.

Rather, we pose the more restricted question: Given the a priori knowledge that Ω is a triaxial ellipsoid, how many measurements must we make to determine the orientation of the object and its principal axis lengths? In the low frequency limit Dassios [6] has shown that six measurements are required and we shall show that the same is true in the high frequency limit. Choosing the six incident wave vectors $(1,0,0)^T$, $(0,1,0)^T$, $(0,0,1)^T$, $(1,1,0)^T$, $(1,0,1)^T$, and $(0,1,1)^T$, we explicitly determine the six coefficients of the symmetric matrix \mathbf{R} from (9a). However, from the definition of \mathbf{R} , (9b), it is evident that $\mathbf{PRP}^T = \mathbf{D}^{-1}$, so that the orthogonal transformation \mathbf{P} simultaneously diagonalizes \mathbf{A} and \mathbf{R} . Therefore, since \mathbf{R} is known, we can numerically compute its eigenvalues and explicitly obtain the lengths of the principal axis. Moreover, since the columns vectors \mathbf{P}_i are the corresponding normalized eigenvectors, we can explicitly determine the Eulerian angles, which define the orientation of Ω , [7] from their components.

IV. Two-Dimensional Result

The analogous two dimensional result is derived in the same fashion and we simply state here the result

$$x = (\cos\phi)d - \sin\phi d' \quad (8a)$$

$$y = (\sin\phi)d + \cos\phi d' \quad (8b)$$

where $d(\phi) = \mathbf{X}_0 \cdot \mathbf{V}_R$, \mathbf{X}_0 is the two-dimensional position vector of the specular point, $\mathbf{V}_R = (\cos\phi, \sin\phi)$ is the negative of the incident wave vector, and the prime denotes differentiation with respect to ϕ .

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Appendix A

We observe (8) that the normal to the ellipsoid at \mathbf{x} is \mathbf{Ax} and consequently that the specular point is the the solution of the equation

$$\mathbf{x}_0 = c\mathbf{v}_R \quad (\text{A1})$$

where c is a fixed constant. Since the specular point, \mathbf{x}_0 , lies on the surface, it follows from (2) and (8) that $d(\phi, \theta) = 1/c$. Using the fact that $\mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{D}$, we deduce from (A1) that

$$\mathbf{x}_0 = d^{-1} \mathbf{P}^T \mathbf{D}^{-1} \mathbf{P} \mathbf{v}_R \quad (\text{A2})$$

where \mathbf{D}^{-1} is the inverse of \mathbf{D} . Taking the scalar product of (A2) with \mathbf{v}_R and recalling the definition of d , (2), we obtain the desired result, (9a).

Figure Caption

Figure 1. Geometry of problem showing the specular point \mathbf{P}^* , the incident and specularly reflected wave numbers, and the fixed origin \mathbf{O} .